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On a doubly nonlinear model for the evolution of damaging in viscoelastic materials

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Abstract

We consider a model describing the evolution of damage in visco-elastic materials, where both the stiffness and the viscosity properties are assumed to degenerate as the damaging is complete. The equation of motion ruling the evolution of macroscopic displacement is hyperbolic. The evolution of the damage parameter is described by a doubly nonlinear parabolic variational inclusion, due to the presence of two maximal monotone graphs involving the phase parameter and its time derivative. Existence of a solution is proved in some subinterval of time in which the damage process is not complete. Uniqueness is established in the case when one of the two monotone graphs is assumed to be Lipschitz continuous.

Keywords: Doubly nonlinear parabolic inclusion; Degenerating parabolic equation; Viscoelasticity; Existence and uniqueness

1. Introduction

In this paper we present some analytical results concerning a model of damage for viscoelastic materials. The system of PDEs we deal with is recovered by the modeling approach proposed by Frémond [6] to describe the phenomenon in terms of continuum mechanics laws. It is known that the phenomenon of damage is caused by microscopic

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actions breaking links in the material. Thus, a good and exhaustive macroscopic theory has to take into account also the effects of these microscopic actions. The idea developed by Frémond consists in generalizing the principle of virtual powers introducing the power of microscopic forces, in duality with microscopic velocities. Consequently, we recover two balance equations, the classical momentum balance and a new equilibrium equation for microscopic forces, which governs the evolution of the damage process. These two equations are complemented by physically meaningful boundary conditions. For the sake of simplicity, we consider an isothermal phenomenon. Hence, the state variables of the system are the symmetric strain tensor $\varepsilon(\mathbf{u})$, depending on the vector of small displacement $\mathbf{u}$, a damage parameter χ, and the gradient $\nabla\chi$. From now on, in regard of simplicity, we deal with a scalar displacement u and replace $\varepsilon(\mathbf{u})$ by ∇u. Concerning the physical meaning of the damage quantity χ, we require that $\chi \in [0, 1]$, letting $\chi = 1$ when the material is undamaged and $\chi = 0$ if the material is completely damaged. Let us point out that the gradient of damage is introduced to take into account the influence of damage at a material point on damage of its neighborhood. For a more detailed presentation of the mechanical features of the model and its applications, see, e.g., [6,8,9]. Hence, the evolving of the damage process depends on dissipative variables, as the damage time derivative χ_t, which is related to microscopic velocities, and also the macroscopic strain rate ∇u_t, as we are considering viscoelastic materials. Even if we do not aim to detail modeling aspects, for the sake of completeness we just stress the main physical features of the problem. The constitutive equations are derived by the free energy Ψ and the pseudo-potential of dissipation Φ, while the balance equations are formally recovered by the generalized principle of virtual power. Now, let us consider a body located in a bounded smooth domain $\Omega \subset \mathbf{R}^3$, whose boundary is Γ, with outward unit normal vector $\mathbf{n}$. In the absence of volumic and surface microscopic actions exerted on the body, the equation governing the damage χ is given in terms of two new interior forces, say B and $\mathbf{H}$, and takes the form

$$B - \operatorname{div}\mathbf{H} = 0. \tag{1.1}$$

These quantities are related to the free energy and the pseudo-potential of dissipation as follows:

$$B = \frac{\partial\Psi}{\partial\chi} + \frac{\partial\Phi}{\partial\chi_t}, \tag{1.2}$$

$$\mathbf{H} = \frac{\partial\Psi}{\partial\nabla\chi}. \tag{1.3}$$

In particular, the dissipative contribution in B, derived by the pseudo-potential Φ, accounts for the evolution of χ. In addition, (1.1) is complemented by a natural boundary assumption

$$\mathbf{H}\cdot\mathbf{n} = 0. \tag{1.4}$$

The classical balance equation for macroscopic movements, in which accelerations are retained, reads in terms of the stress σ

$$u_{tt} - \operatorname{div} \sigma = f, \tag{1.5}$$

where f represents an exterior volumic force applied to the body. We state a non-displacement prescription on the boundary (i.e., an homogeneous Dirichlet boundary condition)

$$u = 0 \quad \text{on } \Gamma. \tag{1.6}$$

We recover σ by deformations, as usual in elasticity relations, and by the strain rate, as we are accounting for viscosity effects, i.e.,

$$\sigma = \frac{\partial \Psi}{\partial \nabla u} + \frac{\partial \Phi}{\partial \nabla u_t}. \tag{1.7}$$

Now, let us make precise the two energy functionals we deal with. We let the free energy be defined by

$$\Psi(\nabla u, \chi, \nabla \chi) = \tfrac{1}{2}\chi|\nabla u|^2 + w(1-\chi) + I_{[0,1]}(\chi) + \tfrac{1}{2}|\nabla \chi|^2, \tag{1.8}$$

where $\frac{1}{2}\chi|\nabla u|^2$ represents the classical elastic contribution in which the rigidity of the material decreases as χ tends to 0, i.e., in the evolution of the damage. Note that, for the sake of simplicity, we have considered the rigidity matrix given by $\chi\,\mathrm{Id}$. The constant $w > 0$ accounts for cohesion in the material, while $I_{[0,1]}$ guarantees the physical consistence of the damage parameter χ, as it makes χ assume values only in the interval $[0, 1]$. Indeed, the indicator function $I_{[0,1]}$ is defined by

$$I_{[0,1]}(\chi) = 0 \quad \text{if } \chi \in [0,1], \qquad I_{[0,1]}(\chi) = +\infty \quad \text{otherwise}. \tag{1.9}$$

Then, the pseudo-potential of dissipation Φ is introduced as a proper, positive, and convex function depending, in general, on dissipative variables (but in our case also on χ)

$$\Phi(\chi_t, \nabla u_t) = \tfrac{1}{2}|\chi_t|^2 + \tfrac{1}{2}\chi|\nabla u_t|^2 + I_{(-\infty,0]}(\chi_t). \tag{1.10}$$

Note that the viscosity contribution in (1.10) depends on a stiffness matrix vanishing when the material is damaged, as it does the elasticity term in (1.8). Finally, the last contribution on the right-hand side of (1.10) represents a constraint on the sign of χ_t, ensuring $\chi_t \leqslant 0$. Indeed, $I_{(-\infty,0)}(\chi_t) = 0$ if $\chi_t \leqslant 0$, while it is equal to $+\infty$ if $\chi_t > 0$. This term characterizes an irreversible evolution of damage, as χ cannot increase, which corresponds to the fact that we are considering materials which cannot reconstruct the interior links broken during the damage process.

Now, we are in the position of writing the complete system

$$u_{tt} - \operatorname{div}(\chi(\nabla u_t + \nabla u)) = f, \tag{1.11}$$

$$\chi_t - \Delta\chi + \partial I_{(-\infty,0]}(\chi_t) + \partial I_{[0,1]}(\chi) \ni w - \frac{1}{2}|\nabla u|^2. \tag{1.12}$$

We recall that $\partial I_{(-\infty,0]}(\chi_t)$ is defined only for $\chi_t \leqslant 0$, and it is $\partial I_{(-\infty,0]}(\chi_t) = \{0\}$ if $\chi_t < 0$ and $\partial I_{(-\infty,0]}(0) = [0,+\infty)$. Analogously, the operator $\partial I_{[0,1]}(\chi)$ turns out to be defined only for $\chi \in [0,1]$ and it is $\partial I_{[0,1]}(\chi) = \{0\}$ if $\chi \in (0,1)$, $\partial I_{[0,1]}(0) = (-\infty, 0]$, and $\partial I_{[0,1]}(1) = [0,+\infty)$. Then, we complete the system by boundary assumptions on Γ

$$\partial_n \chi = 0, \quad u = 0, \tag{1.13}$$

and initial prescriptions in Ω

$$\chi(0) = 1, \quad u(0) = u_0, \quad u_t(0) = v_0. \tag{1.14}$$

Note that the degeneracy of the elasticity and viscosity contributions in (1.11) as $\chi \searrow 0$ comes from the fact that the material loses its physical properties in damaging. However, macroscopic deformations are involved as a quadratic source of damage in the equation governing the evolution of χ (1.12). In particular, it results that this quadratic contribution cannot be controlled once the material is damaged. This fact, combined with the presence of a double nonlinearity in (1.12) involving monotone constraints, makes the system very difficult to be solved in the whole time interval where we are investigating the phenomenon. Thus, following the idea exploited in [4,7], we restrict ourselves to consider the evolution of damage when the damage parameter χ remains strictly greater then a positive constant δ, i.e., when the material is not completely damaged and retains some stiffness and viscosity properties. Differently from [4,7], where the term $\partial I_{[0,1]}(\chi)$ is neglected in the relation corresponding to (1.12), we keep such term and even assume that it can be replaced by a more general maximal monotone graph β, in regard of different dynamics. This yields the doubly nonlinear character of (1.12).

Before entering the details of the subject of the paper, let us briefly recall some analytical results in the literature related to the Frémond damage model. From the point of view of applications, numerical results show that the system provides a behavior of the damaged materials in accordance with experiments (cf. [6] and references therein). On the contrary, analytical results have been obtained only for some simplified version of the system or in the one-dimensional case (cf. [7,8]). We point out that the investigation of the one-dimensional model shows local-in-time existence and uniqueness of solutions. The local character of these results is mainly due to the presence of a quadratic source of damage which becomes unbounded in the evolving of the damage. The three-dimensional model has been investigated in [4] from the point of view of the existence and uniqueness of the solution in a finite time interval in which $\chi \geqslant \delta > 0$. More precisely, by considering also some dissipative effects on the gradient of damage, but neglecting any viscosity contributions for macroscopic deformations in the momentum balance, the authors show that for any fixed $\delta > 0$ there exists a time $\widehat{t}$ such that the problem admits a solution in $(0, \widehat{t})$, with $\chi \geqslant \delta$. Moreover, uniqueness is proved for any solution with $\chi > 0$. Finally, we mention the paper [2] in which the authors investigate relations between macroscopic deformations and the microscopic motions which are responsible for damage assuming some viscosity effects. However, our situation is fairly different, as we cannot recover a uniform bound for deformations by the presence of the viscosity since this term degenerates during the damage process and does not help to provide more regularity on deformations. In particular, the well-posedness of the complete model in any time interval remains an open question and it seems that the model itself has to be improved, e.g. by adding some constitutive relations to characterize the behavior of a completely damaged material.

Here is the outline of the paper. In Section 2 we state our hypotheses on data and list our main results. Namely, we introduce a non-degenerate version of the system by replacing χ in (1.11) by its truncation at the level $\delta \in (0, 1)$. In Section 3 we investigate the truncated problem and prove existence of a solution. This result is obtained regularizing the monotone constraints on the internal variables and exploiting an a priori estimates-passage to the limit technique, joint with a fixed point argument. In Section 4 we show that, at least for small times, the component χ of the solution to the truncated problem stays above the barrier δ a.e. in Ω; hence, it also solves the original (non-truncated) problem. Finally, in Section 5 uniqueness of the solution is shown by using contracting estimates.

2. Main results

First of all, we recall that Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\Gamma = \partial\Omega$. We also set $H := L^2(\Omega)$, $V := H^1(\Omega)$ (endowed with usual scalar products), in order that, identifying H with H', we get the Hilbert triplet $V \subset H \subset V'$. We denote by $\| \cdot \|_E$ the norm in the generic normed spaces E and E^3. In particular, $\| \cdot \|_V$ will stand both for the norm in V and in its closed subspace H_0^1. Moreover, we denote by $(\cdot, \cdot)$ the scalar product in H and by $\langle \cdot, \cdot \rangle$ the duality product between the space E and its topological dual E'. Letting $\chi : \Omega \to [0, 1]$ be a measurable function, we consider the (possibly degenerate) elliptic operator $-\mathrm{div}(\chi\nabla\cdot)$ which maps

$$\begin{aligned} &H_0^1(\Omega) \to H^{-1}(\Omega), \qquad v \mapsto -\mathrm{div}(\chi\nabla v), \\ &\text{where } \langle -\mathrm{div}(\chi\nabla v), z\rangle = \big(\chi\nabla v, \nabla z\big) \qquad \forall z \in H_0^1(\Omega). \end{aligned} \tag{2.1}$$

Analogously, we introduce the realization of the Laplace operator with homogeneous Neumann boundary conditions as

$$\mathcal{B} : V \to V', \qquad \langle \mathcal{B}u, v\rangle = \big(\nabla u, \nabla v\big) \quad \forall u, v \in V. \tag{2.2}$$

We also define

$$W := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma\}, \tag{2.3}$$

which is a closed subspace of $H^2(\Omega)$ by continuity of the trace operator.

Remark 2.1. Let us observe that if u and χ belong to $H^2(\Omega)$, then there hold

$$-\mathrm{div}(\chi\nabla u) \in H, \quad \text{and} \quad -\mathrm{div}(\chi\nabla u) = -\chi\Delta u - \nabla\chi \cdot \nabla u. \tag{2.4}$$

This fact can be proved by means of an approximation-density procedure. Thus, in such a regularity framework, the term $-\mathrm{div}(\chi\nabla v)$ makes sense in H, hence almost everywhere in Ω. Analogously, also the term $\mathcal{B}u$ in (2.2) can be understood as an L^2-function as we have $u \in W$.

Looking back at Eqs. (1.11)–(1.12), we assume the following hypotheses:

$$f \in L^2(0, T; H), \tag{2.5}$$

$$w > 0, \tag{2.6}$$

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \quad v_0 \in H_0^1(\Omega), \tag{2.7}$$

$$\alpha \subset \mathbb{R} \times \mathbb{R} \text{ maximal monotone graph given by } \alpha = \partial I_{(-\infty,0]}, \tag{2.8}$$

$$\beta \subset \mathbb{R} \times \mathbb{R} \text{ maximal monotone graph such that } [0, 1] \subset D(\beta),\ 0 \in \beta(0). \tag{2.9}$$

Since any maximal monotone graph in $\mathbb{R}^2$ is cyclically monotone (cf., e.g., [5, pp. 38, 43]), by (2.9) it follows that there exists a convex, lower semicontinuous and proper function $\phi : \mathbb{R} \to [0, +\infty]$ such that $\beta = \partial\phi$, with $0 = \min \phi = \phi(0)$. Also, for the sake of simplicity, we set $\psi := I_{(-\infty,0]}$ and $\alpha := \partial\psi$. Finally, in order to reformulate our problem (1.11)–(1.12) in the abstract setting of the above Hilbert spaces, we introduce the functional induced by ϕ on H, namely (see [5, p. 47])

$$\Phi(v) := \begin{cases} \int_\Omega \phi(v) & \text{if } \phi(v) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.10}$$

It is a well-known fact that, under the above assumptions on ϕ, Φ is still a proper, l.s.c., and convex functional mapping H into $[0, +\infty]$, such that its subdifferential $\partial\Phi$ acts as a maximal monotone operator in $H \times H$. Furthermore, for $z, v \in H$, we have $v \in \partial\Phi(z)$ if and only if $v(x) \in \partial\phi(z(x))$ for a.a. $x \in \Omega$. In a similar way we construct Ψ as the realization in H of the proper, l.s.c., and convex function $\psi = I_{(-\infty,0]}$. For the sake of simplicity and with a slight abuse of notation, we will write α for $\partial\Psi$ and β for $\partial\Phi$.

We can now state the main result of this paper.

Theorem 2.2. *Let assumptions* (2.5)–(2.9) *hold. Then, for each* (*small*) *positive constant* δ, *there exist* $\hat{t}(\delta) \in [0, T]$ (*depending on* δ), *and a quadruple* (χ, ξ, η, u) *with regularity*

$$\chi \in H^1(0, \hat{t}; V) \cap L^\infty(0, \hat{t}; W), \tag{2.11}$$

$$\xi \in L^\infty(0, \hat{t}; H), \tag{2.12}$$

$$\eta \in L^\infty(0, \hat{t}; H), \tag{2.13}$$

$$u \in H^2(0, \hat{t}; H) \cap W^{1,\infty}(0, \hat{t}; H_0^1(\Omega)) \cap H^1(0, \hat{t}; H^2(\Omega)), \tag{2.14}$$

$$\chi \geqslant \delta \quad \textit{in } \Omega \times [0, \hat{t}], \tag{2.15}$$

and fulfilling a.e. in $\Omega \times (0, \hat{t})$

$$\partial_t \chi + \xi + \mathcal{B}\chi + \eta = w - \frac{|\nabla u|^2}{2}, \tag{2.16}$$

$$\xi \in \alpha(\partial_t \chi), \tag{2.17}$$

$$\eta \in \beta(\chi), \tag{2.18}$$

$$\partial_{tt} u - \operatorname{div}\big(\chi \nabla (u + \partial_t u)\big) = f, \tag{2.19}$$

and the initial conditions

$$\chi(\cdot, 0) = 1, \quad u(\cdot, 0) = u_0, \quad \partial_t u(\cdot, 0) = v_0, \tag{2.20}$$

a.e. in Ω.

Remark 2.3. Note that, taking the third of (2.14) into account, a comparison in (2.16) gives a further regularity property for χ, namely $\chi \in W^{1,\infty}(0, \hat{t}; H)$.

If we assume β Lipschitz continuous, we also have the following uniqueness result:

Theorem 2.4. *Suppose we are given data satisfying* (2.5)–(2.9), *a positive time* $\hat{t}$ *and* $\delta \in (0, 1)$. *Then, if the restriction of* β *to the interval* $[\delta, 1]$ *is Lipschitz continuous and if* $(\chi_1, \xi_1, \eta_1, u_1)$ *and* $(\chi_2, \xi_2, \eta_2, u_2)$ *fulfill* (2.11)–(2.20), *then there holds*

$$(\chi_1, \xi_1, \eta_1, u_1) = (\chi_2, \xi_2, \eta_2, u_2) \quad \textit{a.e. in } \Omega \times (0, \hat{t}). \tag{2.21}$$

The proof of Theorem 2.2 will be carried out by exploiting a truncation procedure. Thus, we introduce a regularized version of (2.16)–(2.19) obtained by truncating the elastic coefficient in (2.19) by means of the truncation operator T_δ given by $T_\delta(r) := \max\{r, \delta\}$, with $\delta \in (0, 1)$. The local existence result for the truncated system reads as follows:

Theorem 2.5. *Under assumptions* (2.5)–(2.9), *for any* $\delta \in (0,1)$ *there exist* $T_0 \in (0,T]$ *and a quadruple* (χ, ξ, η, u) *with*

$$\chi \in H^1(0,T_0;V) \cap L^\infty(0,T_0;W), \tag{2.22}$$

$$\xi \in L^\infty(0,T_0;H), \tag{2.23}$$

$$\eta \in L^\infty(0,T_0;H), \tag{2.24}$$

$$u \in H^2(0,T_0;H) \cap W^{1,\infty}(0,T_0;H_0^1(\Omega)) \cap H^1(0,T_0;H^2(\Omega)), \tag{2.25}$$

and fulfilling a.e. in $\Omega \times (0,T_0)$ *the equations*

$$\partial_t \chi + \xi + \mathcal{B}\chi + \eta = w - \frac{|\nabla u|^2}{2}, \tag{2.26}$$

$$\xi \in \alpha(\partial_t \chi), \tag{2.27}$$

$$\eta \in \beta(\chi), \tag{2.28}$$

$$\partial_{tt} u - \operatorname{div}\big(T_\delta(\chi)\nabla(u + \partial_t u)\big) = f, \tag{2.29}$$

and the initial conditions

$$\chi(\cdot,0) = 1, \quad u(\cdot,0) = u_0, \quad \partial_t u(\cdot,0) = v_0, \tag{2.30}$$

a.e in Ω.

3. Proof of Theorem 2.5

In this section we detail the proof of Theorem 2.5 by means of the Schauder fixed-point argument. To this end, let us first define the correct space to exploit this procedure. Given $R, T_0 > 0$ (to be chosen later) we set

$$\mathcal{U} = \left\{ v \in H^1(0,T_0;W_0^{1,4}(\Omega)) \ : \| v \|_4 \leqslant R \right\}, \tag{3.1}$$

where $\| \cdot \|_4$ stands for the usual norm in $H^1(0,T_0;W_0^{1,4}(\Omega))$. Then, we construct an operator $\mathcal{S}$ which will be shown to map $\mathcal{U}$ into itself. Actually, in order to define properly such map (see (3.13) below), we need to regularize Eq. (2.16) by replacing the nonlinear multivalued operator β with its Yosida approximation β^ε, with $\varepsilon > 0$ intended to go to 0 in the limit. The passage to the limit procedure will be carefully investigated in the forthcoming Section 3.4. Concerning the properties of the function β^ε, we remark that it turns out to be a globally Lipschitz continuous mapping with Lipschitz constant $\leqslant \varepsilon^{-1}$ (see [5, Proposition 2.6, p. 28]). To simplify the notation, from now on we denote by $m(R,T)$ (or $m_i(R,T)$, $i = 0,1,2,\ldots$) some possibly different positive functions depending on the data of the problem and $\delta \in (0,1)$ (but not on ε) which do not go to ∞ when one, or both, their arguments R, T go to 0.

The construction of the operator $\mathcal{S}$ will be carried out in two steps.

Problem 3.1 (*First step*). For given $\bar{u} \in \mathcal{U}$, find $(\chi := \mathcal{S}_1(\bar{u}), \xi) : \Omega \times (0, T_0) \to \mathbb{R}^2$ solving the system

$$\xi + \partial_t \chi + \mathcal{B}\chi + \beta^\varepsilon(\chi) = w - \frac{|\nabla \bar{u}|^2}{2}, \tag{3.2}$$

$$\xi \in \alpha(\partial_t \chi), \tag{3.3}$$

$$\chi(0) = 1. \tag{3.4}$$

Note that in the above statement we do not need the symbol η to denote $\beta^\varepsilon(\chi)$, since actually β^ε is single valued. Our result for Problem 3.1 can be stated as follows:

Lemma 3.2. *For any* $R, T_0 > 0$, $\bar{u} \in \mathcal{U}$, *and* $\delta \in (0,1)$, *there exists a unique couple* $(\chi = \mathcal{S}_1(\bar{u}), \xi)$ *fulfilling* (3.2)–(3.4) *and such that*

$$\chi \in H^1(0, T_0, V) \cap L^\infty(0, T_0, W), \tag{3.5}$$

$$\xi \in L^\infty(0, T_0; H), \tag{3.6}$$

$$\| \chi \|^2_{H^1(0,T_0;V)\cap L^\infty(0,T_0;W)} \leqslant m_0(R, T_0). \tag{3.7}$$

The second step of our fixed point argument can be described by introducing the following:

Problem 3.3 (*Second step*). For given

$$\bar{\chi} \in \mathcal{X} = \mathcal{X}(R, T_0) := \Big\{ \bar{\chi} \in H^1(0, T_0; V) \cap L^\infty(0, T_0; W) : \| \bar{\chi} \|^2_{H^1(0,T_0;V)\cap L^\infty(0,T_0;W)} \leqslant m_0(R, T_0) \Big\}, \tag{3.8}$$

find $u := \mathcal{S}_2(\chi) : \Omega \times (0, T_0) \to \mathbb{R}$ solving

$$\partial_{tt} u - \operatorname{div}\big(T_\delta(\bar{\chi}) \nabla (u + \partial_t u)\big) = f \quad \text{a.e. in } \Omega \times (0, T_0),$$

$$u = 0 \quad \text{a.e. on } \Gamma \times (0, T_0), \tag{3.9}$$

$$u(0) = u_0, \quad u_t(0) = v_0 \quad \text{a.e. in } \Omega. \tag{3.10}$$

The existence-regularity result related to Problem 3.3 is as follows:

Lemma 3.4. *For any $R > 0$ and $\delta \in (0, 1)$, there exists $T_0 = T_0(\delta) \in (0, T]$, depending on δ, such that, for any $\bar{\chi} \in \mathcal{X}$, Problem* 3.3 *has one and only one solution $u = \mathcal{S}_2(\bar{\chi})$ satisfying*

$$u \in H^2(0, T_0; H) \cap W^{1,\infty}(0, T_0; H_0^1(\Omega)) \cap H^1(0, T_0; H^2(\Omega)), \tag{3.11}$$

$$\| u \|_4 \leqslant R. \tag{3.12}$$

Clearly, these two lemmata lead to the construction of the desired operator $\mathcal{S}$, defined as the composition $\mathcal{S}_2 \circ \mathcal{S}_1$. The properties of $\mathcal{S}$ are stated by the following:

Theorem 3.5. *Given $R > 0$ and $\delta \in (0, 1)$, there exists a time $T_0 \in (0, T]$ such that*

$$\mathcal{S} : \mathcal{U}(R, T_0) \to \mathcal{U}(R, T_0) \quad \textit{is well defined}, \tag{3.13}$$

$$\mathcal{S} \quad \textit{is continuous with respect to the norm} \quad \| \cdot \|_4, \tag{3.14}$$

$$\mathcal{S} \quad \textit{is a compact map}, \tag{3.15}$$

The rest of this section is devoted to the proofs of Lemmas 3.2, 3.4 and of Theorem 3.5. The latter Theorem 3.5 guarantees in particular that $\mathcal{S}$ fulfills the assumptions of Schauder's fixed point Theorem. Hence, $\chi = \mathcal{S}_1(u)$, where u is a fixed point of $\mathcal{S}$, is a solution to Problem 3.1 for $\bar{u} = u$ and u is a solution to Problem 3.3 where $\bar{\chi} = \mathcal{S}_1(u)$.

3.1. Proof of Lemma 3.2

Let us consider $\bar{u} \in \mathcal{U}$; then, setting $g(x, t) := -|\nabla \bar{u}(x, t)|^2/2$ we notice that $g \in H^1(0, T_0; H)$ and there exists a positive constant C depending only on the embeddings $H^1(0, T_0; W_0^{1,4}(\Omega)) \hookrightarrow L^\infty(0, T_0; W_0^{1,4}(\Omega)) \hookrightarrow L^4(0, T_0; W_0^{1,4}(\Omega))$ such that

$$\| g \|_{H^1(0,T_0;H)} \leqslant C R^2. \tag{3.16}$$

Now, remarking that g is assigned fulfilling (3.16) and that the nonlinearity β^ε is Lipschitz continuous, we aim to find a sufficiently regular solution to the differential inclusion

$$\alpha(\partial_t \chi) + \partial_t \chi + \mathcal{B}\chi + \beta^\varepsilon(\chi) \ni w + g \tag{3.17}$$

combined with condition (3.4). More precisely, we look for a solution satisfying properties (3.5)–(3.7). A possible way to address this problem is to substitute the nonlinear operator α with its Yosida approximation α_λ for $\lambda > 0$ intended to go to zero in the limit and solve the regularized problem by means of a time discretization scheme. We prefer not to go into the details of this argument since it is quite close to the investigation devised by Bonfanti et al. [3]. However, we point out that this procedure entails existence and uniqueness of a solution χ_λ to (the λ-regularized version of) (3.17) together with (3.4) which satisfies

$$\chi_\lambda \in H^1(0, T_0; V) \cap L^\infty(0, T_0; W). \tag{3.18}$$

Then, assuming to have such χ_λ, let us perform some estimates in order to remove the approximation in λ. Actually, it is worthwhile noting that the estimates we derive will be independent of both λ and ε. Moreover, from now, and up to the end of this paper, the symbol C will be used to denote some positive constants (possibly different from each other) appearing in the computations and only depending on data, but not on T_0, δ, R. If the constant depends on additional parameters (e.g., T_0), we will indicate this by using a symbol like $C(\cdot)$. Moreover, we denote by, e.g., c_σ possible different positive constants allowed to depend in addition on positive (small) parameters (here denoted by σ). In particular, we make use of the Young inequality in the form

$$ab \leqslant \sigma a^2 + c_\sigma b^2 \quad \forall a, b \in R,\ \sigma > 0. \tag{3.19}$$

For simplicity, the superscript ε will be temporarily omitted in denoting the solution.

First estimate: Test (3.17) by $\partial_t \chi_\lambda$ and integrate the resulting relation on $[0, t]$ with $t \leqslant T_0$. The monotonicity of α_λ, (2.10), (2.20), (3.19), and Sobolev's embeddings entail, for any $t \in [0, T_0]$, the following inequality:

$$\begin{aligned} &\frac{1}{2}\int_0^t \| \partial_t \chi_\lambda(s) \|_H^2 \, ds + \frac{1}{2} \| \nabla \chi_\lambda(t) \|_H^2 + \Phi(\chi_\lambda(t)) \\ &\quad \leqslant w^2 T_0 |\Omega| + \Phi(1) + C \| \bar{u} \|^4_{H^1(0,T_0; W^{1,4}(\Omega))} . \end{aligned} \tag{3.20}$$

Let us remark that $\Phi(1) < +\infty$ as $1 \in D(\beta)$ (cf. (2.9)).

Second estimate: Test (3.17) by $\partial_t(\mathcal{B}\chi_\lambda + \beta(\chi_\lambda))$ and integrate on the time interval $[0, t]$ with $t \leqslant T_0$. We obtain

$$\begin{aligned} &\frac{1}{2} \| \big(\mathcal{B}\chi_\lambda + \beta(\chi_\lambda)\big)(t) \|_H^2 + \int_0^t \| \nabla \partial_t \chi_\lambda(s) \|_H^2 \, ds \\ &\quad \leqslant \frac{1}{2} \| \beta(1) \|_H^2 + \int_0^t \left(w - \frac{|\nabla \bar{u}|^2}{2}, \partial_t\big(\mathcal{B}\chi_\lambda + \beta(\chi_\lambda)\big) \right)(s)\, ds \\ &\quad =: \frac{1}{2} \| \beta(1) \|^2 + I_1(t), \end{aligned} \tag{3.21}$$

thanks to the monotonicity of the operators α_λ, β and to (3.19). Our aim is now to bound the right-hand side of (3.21). In this direction, an integration by parts with respect to time gives

$$I_1(t) = \left(w - \frac{|\nabla \bar{u}(t)|^2}{2}, \mathcal{B}\chi_\lambda(t) + \beta(\chi_\lambda(t))\right) - \left(w - \frac{|\nabla \bar{u}_0|^2}{2}, \beta(1)\right)$$
$$+ \int_0^t \big(\mathcal{B}\chi_\lambda + \beta(\chi_\lambda), \nabla\bar{u}\nabla\partial_t\bar{u}\big)(s)\, ds := I_2(t) + I_3(t) + I_4(t). \quad (3.22)$$

Thus, thanks to (3.19), Poincaré's inequality, and Sobolev's embeddings, we can deduce

$$I_2(t) \leqslant \tfrac{1}{4} \parallel \big(\mathcal{B}\chi_\lambda + \beta(\chi_\lambda)\big)(t) \parallel_H^2 + 2w^2|\Omega| + C \parallel \bar{u} \parallel^4_{H^1(0,T_0;W_0^{1,4}(\Omega))}, \quad (3.23)$$

$$I_3(t) \leqslant w^2|\Omega| + C \parallel \bar{u}_0 \parallel^4_{W_0^{1,4}(\Omega)} + \tfrac{1}{2} \parallel \beta(1) \parallel_H^2, \quad (3.24)$$

$$I_4(t) \leqslant C\left(\int_0^t \parallel \big(\mathcal{B}\chi_\lambda + \beta(\chi_\lambda)\big)(s) \parallel_H^2 \parallel \bar{u}(s) \parallel^2_{W_0^{1,4}(\Omega)} ds\right.$$
$$\left. + \int_0^t \parallel \partial_t\bar{u}(s) \parallel^2_{W_0^{1,4}(\Omega)} ds\right). \quad (3.25)$$

Now, summing (3.20) and (3.21), collecting (3.23)–(3.25), and recalling (3.1), for any $t \in [0, T_0]$ we get

$$\frac{1}{4} \parallel (\mathcal{B}\chi_\lambda + \beta(\chi_\lambda))(t) \parallel_H^2 + \frac{1}{2}\int_0^t \parallel \partial_t\chi_\lambda(s) \parallel_V^2 ds + \frac{1}{2} \parallel \nabla\chi_\lambda(t) \parallel_H^2 + \Phi(\chi_\lambda(t))$$
$$\leqslant w^2|\Omega|(3 + T_0) + CR^4 + \parallel \beta(1) \parallel_H^2 + \Phi(1)$$
$$+ C\left(\int_0^t \parallel (\mathcal{B}\chi_\lambda + \beta(\chi_\lambda))(s) \parallel_H^2 \parallel \bar{u}(s) \parallel^2_{W_0^{1,4}(\Omega)} ds\right.$$
$$\left. + \int_0^t \parallel \partial_t\bar{u}(s) \parallel^2_{W_0^{1,4}(\Omega)} ds\right). \quad (3.26)$$

Next, we can apply the Gronwall Lemma and find a positive constant $m_1(R, T_0)$ such that (cf. also (3.1))

$$\parallel \big(\mathcal{B}\chi_\lambda + \beta(\chi_\lambda)\big)(t) \parallel_H^2 + \int_0^t \parallel \partial_t\chi_\lambda(s) \parallel_V^2 ds + \parallel \nabla\chi_\lambda(t) \parallel_H^2$$
$$\leqslant m_1(R, T_0) \quad \forall t \in [0, T_0]. \quad (3.27)$$

Then, noting that, $\forall t \in [0, T_0]$ and a.e. $x \in \Omega$ it is

$$|\chi_\lambda(x,t)| \leqslant 1 + \int_0^t |\partial_t\chi_\lambda(x,s)|\, ds,$$

so that

$$\| \chi_\lambda(t) \|_H^2 \leqslant 2|\Omega| + 2T_0 \int_0^t \| \partial_t \chi_\lambda(s) \|_H^2 \, ds \tag{3.28}$$

(and an analogous relation for the gradient), we easily get from (3.27)–(3.28)

$$\begin{aligned}
\| \chi_\lambda \|^2_{H^1(0,T_0;V)} &\leqslant T_0 \Big(\| \chi_\lambda \|^2_{L^\infty(0,T_0;H)} + \| \nabla \chi_\lambda \|^2_{L^\infty(0,T_0;H)} \Big) \\
&\quad + \| \partial_t \chi_\lambda \|^2_{L^2(0,T_0;H)} + \| \nabla \partial_t \chi_\lambda \|^2_{L^2(0,T_0;H)} \\
&\leqslant T_0 \Big(4|\Omega| + 2T_0 \big(\| \partial_t \chi_\lambda \|^2_{L^2(0,T_0;H)} + \| \nabla \partial_t \chi_\lambda \|^2_{L^2(0,T_0;H)} \big) \Big) \\
&\quad + \| \partial_t \chi_\lambda \|^2_{L^2(0,T_0;H)} + \| \nabla \partial_t \chi_\lambda \|^2_{L^2(0,T_0;H)} \\
&\leqslant 4T_0|\Omega| + (1 + 2T_0^2) m_1(R, T_0) =: m_2(R, T_0).
\end{aligned} \tag{3.29}$$

Next, let us point out that the monotonicity of β gives, for almost any $t \in [0, T_0]$,

$$\| (\mathcal{B}\chi_\lambda + \beta(\chi_\lambda))(t) \|_H^2 \geqslant \| \mathcal{B}\chi_\lambda(t) \|_H^2 + \| \beta(\chi_\lambda(t)) \|_H^2 . \tag{3.30}$$

Thus, from (3.27), (3.29), and standard elliptic regularity results, we get

$$\| \chi_\lambda \|^2_{L^\infty(0,T_0;W)} \leqslant m_3(R, T_0). \tag{3.31}$$

Finally, combining (3.27), (3.29) and (3.31), one concludes that any solution of the λ-regularized problem satisfies (3.7). Clearly, this upper bound will be conserved provided that we are able to pass to the limit as $\lambda \searrow 0$. Actually, by means of (3.7) we have a limit function χ (candidate to be the solution to Problem 1) such that (up to the extraction of a suitable subsequence of $\lambda \searrow 0$, not relabeled)

$$\chi_\lambda \to \chi \quad \text{weakly star in } H^1(0, T_0; V) \cap L^\infty(0, T_0; W). \tag{3.32}$$

Now, by the generalized Aubin compactness Lemma (see [12, Corollary 4]), we have

$$\chi_\lambda \to \chi \quad \text{strongly in } C^0([0, T_0]; V). \tag{3.33}$$

We recall that for the moment we are investigating only the λ-limit, so that ε is actually a fixed parameter. The latter convergence (3.33), combined with the Lipschitz continuity of the Yosida approximation β^ε of β, gives

$$\beta(\chi_\lambda) \to \beta(\chi) \quad \text{strongly in } C^0([0, T_0]; H). \tag{3.34}$$

Observing now that Eq. (3.17) can be rewritten in the equivalent form

$$(Id+\alpha_\lambda)(\partial_t\chi_\lambda) = w - \frac{|\nabla\bar{u}|^2}{2} - \mathcal{B}\chi_\lambda - \beta(\chi_\lambda), \tag{3.35}$$

a comparison of terms on the right-hand side of (3.35) yields

$$(Id+\alpha_\lambda)(\partial_t\chi_\lambda) \to \zeta := w - \frac{|\nabla\bar{u}|^2}{2} - \mathcal{B}\chi - \beta(\chi) \quad \text{weakly star in } L^\infty(0,T_0;H). \tag{3.36}$$

Thus, passing to the limit, we get that relation (3.2) actually holds with ζ in place of $\xi+\partial_t\chi$. Our next aim is to give an interpretation of ζ in terms of the operator α, that is to prove $\zeta \in (Id+\alpha)(\partial_t\chi)$ or, equivalently, (3.3) as we set $\xi := \zeta - \partial_t\chi$. To this aim, we first observe that $(Id+\alpha_\lambda)^{-1}$ is Lipschitz continuous uniformly with respect to λ. Thus

$$\partial_t\chi_\lambda = (\alpha_\lambda + Id)^{-1}\Big(w - \frac{|\nabla\bar{u}|^2}{2} - \mathcal{B}\chi_\lambda - \beta(\chi_\lambda)\Big),$$

converges weakly star in $L^\infty(0,T_0;H)$ to $\partial_t\chi$. Hence, we can exploit a semicontinuity-comparison procedure (see [5, Proposition 2.5, p. 27]) to eventually characterize ζ. Namely, let us test (3.17) by $\partial_t\chi_\lambda$ and integrate over $(0,T_0)$. We get

$$\begin{aligned}
&\int_0^{T_0} \| \partial_t\chi_\lambda(s) \|_H^2 \, ds + \int_0^{T_0} \big(\alpha_\lambda(\partial_t\chi_\lambda), \partial_t\chi_\lambda\big)(s)\,ds \\
&\quad = \int_0^{T_0} \Big(w - \frac{|\nabla\bar{u}|^2}{2}, \partial_t\chi_\lambda\Big)(s)\,ds - \int_0^{T_0} \big(\beta(\chi_\lambda), \partial_t\chi_\lambda\big)(s)\,ds \\
&\qquad - \frac{1}{2} \| \nabla\chi_\lambda(T_0) \|_H^2 .
\end{aligned} \tag{3.37}$$

By (3.32) and (3.33), we have

$$\begin{aligned}
&\lim_{\lambda\searrow 0}\Big[-\frac{1}{2} \| \nabla\chi_\lambda(T_0) \|_H^2 + \int_0^{T_0} \Big(w - \frac{|\nabla\bar{u}|^2}{2}, \partial_t\chi_\lambda\Big)(s)\,ds\Big] \\
&\quad = -\frac{1}{2} \| \nabla\chi(T_0) \|_H^2 + \int_0^{T_0} \Big(w - \frac{|\nabla\bar{u}|^2}{2}, \partial_t\chi\Big)(s)\,ds.
\end{aligned} \tag{3.38}$$

Moreover, by (3.32) and (3.34), it is

$$\lim_{\lambda\searrow 0}\Big[-\int_0^{T_0} \big(\beta(\chi_\lambda), \partial_t\chi_\lambda\big)(s)\,ds\Big] = -\int_0^{T_0} \big(\beta(\chi), \partial_t\chi\big)(s)\,ds. \tag{3.39}$$

Now, let us take the lim sup of (3.37) and compare it with the limit relation tested by $\partial_t \chi$ and integrated in time: on account of (3.38)–(3.39), we readily obtain that

$$\limsup_{\lambda \searrow 0} \int_0^{T_0} \big((Id + \alpha_\lambda)(\partial_t \chi_\lambda), \partial_t \chi_\lambda\big)(s)\, ds \leqslant \int_0^{T_0} \big(\zeta, \partial_t \chi\big)(s)\, ds, \tag{3.40}$$

which gives the desired identification, i.e. $\zeta \in (Id + \alpha)(\partial_t \chi)$ (cf. [5, Proposition 2.5, p. 27]). Then, letting $\xi := \zeta - \partial_t \chi$, it is a standard matter to find (3.3). To conclude the proof, it remains to show that the couple (χ, ξ) solving Problem 3.1 is unique; this entails that the whole sequence $(\chi_\lambda, \xi_\lambda)$ converges to (χ, ξ). To this end, we consider two solutions (χ_1, ξ_1), (χ_2, ξ_2) with $\xi_i \in \alpha(\partial_t \chi_i), i = 1, 2, i = 1, 2$, take the difference between the corresponding equations (3.2) and test it by $\partial_t(\chi_1 - \chi_2)$. After an integration in time and recalling the Lipschitz continuity of β^ε, we obtain (we set, for simplicity of notation, $\chi := \chi_1 - \chi_2$)

$$\| \partial_t \chi \|^2_{L^2(0,t;H)} + \frac{1}{2} \| \nabla \chi(t) \|^2_H \leqslant \frac{1}{\varepsilon} \int_0^t \| \chi(s) \|_H \| \partial_t \chi(s) \|_H \; ds. \tag{3.41}$$

In order to recover the full *V*-norm of χ on the left-hand side, we add to (3.41) the inequality

$$\sigma \| \chi(t) \|^2_H \leqslant \sigma T_0 \int_0^t \| \partial_t \chi(s) \|^2_H \; ds \quad \text{for a.e. } t \in (0, T_0). \tag{3.42}$$

Next, exploiting Young's inequality in the form (3.19) in order to split the right-hand side of (3.41) and taking $\sigma < 1/2T_0$, by the Gronwall Lemma we get $\chi_1 = \chi_2$ a.e. in $\Omega \times (0, T_0)$. Thus, a comparison in (3.2), implies $\xi_1 = \xi_2$ a.e., which concludes the proof.

3.2. Proof of Lemma 3.4

Now, let us fix $\bar{\chi}$ (with the regularity prescribed by (3.8)) in Eq. (3.9). Since we are going to take $\bar{\chi} = \mathcal{S}_1(\bar{u}) = \chi$ at the end, we shall write already from the beginning χ in place of $\bar{\chi}$, for simplicity. Then, since $\delta \leqslant T_\delta(\chi) \leqslant 1$ almost everywhere, the family of operators (depending on $t \in [0, T_0]$)

$$H^1_0(\Omega) \to H^{-1}(\Omega), \qquad v \to -\mathrm{div}\big(T_\delta(\chi(t))\nabla v\big) \tag{3.43}$$

is uniformly strongly elliptic w.r.t. $t \in [0, T_0]$. Thus, by (2.5), (3.8) and standard well-posedness theorems, there exists a unique solution to Problem 3.3 with the prescribed regularity. Let us now perform the quantitative estimates relating the regularity of *u* to

δ and to suitable norms of χ. Namely, let us test (2.29) by $\partial_t(u-\Delta u)$ and integrate it in time: the considerations above entail

$$\frac{1}{2}\parallel \partial_t u(t) \parallel_V^2 + \frac{\delta}{2}\int_0^t \parallel \Delta\partial_t u(s) \parallel_H^2 \, ds \leqslant \sum_{i=5}^{10} I_i(t), \tag{3.44}$$

where

$$I_5(t) := \frac{1}{2}\left(\parallel v_0 \parallel_V^2 + \left(1+\frac{1}{\delta}\right)\int_0^t \parallel f(s) \parallel_H^2 \, ds + \int_0^t \parallel u_t(s) \parallel_H^2 \, ds\right),$$

$$I_6(t) := \int_0^t \big(T_\delta'(\chi)\nabla\chi(\nabla u + \nabla\partial_t u), \partial_t u\big)(s)\, ds,$$

$$I_7(t) := \int_0^t \big(T_\delta(\chi)(\Delta u + \Delta\partial_t u), \partial_t u\big)(s)\, ds,$$

$$I_8(t) := -\int_0^t \big(T_\delta(\chi)\Delta u, \Delta\partial_t u\big)(s)\, ds,$$

$$I_9(t) := -\int_0^t \big(T_\delta'(\chi)\nabla\chi\nabla u, \Delta\partial_t u\big)(s)\, ds,$$

$$I_{10}(t) := -\int_0^t \big(T_\delta'(\chi)\nabla\chi\nabla\partial_t u, \Delta\partial_t u\big)(s)\, ds.$$

Our next goal is to estimate the terms $I_i(t)$ for $i = 6, \ldots, 10$. Let us start with I_6. Thanks to Young's inequality, Sobolev's embeddings, and the definition of T_δ we have

$$I_6(t) \leqslant C \parallel \chi \parallel_{L^\infty(0,T_0;W)} \left(\int_0^t \parallel \partial_t u(s) \parallel_V^2 \, ds + \int_0^t \parallel \nabla u(s) \parallel_H^2 \, ds\right). \tag{3.45}$$

Referring to the latter bound, I_7, I_8 and I_9 (cf. also the definition and the uniform bound of the operator T_δ) will be treated similarly. Namely, we have

$$\begin{aligned} I_7(t) \leqslant{}& c_\sigma \int_0^t \parallel \partial_t u(s) \parallel_H^2 \, ds + c_\sigma \int_0^t \parallel \Delta u(s) \parallel_H^2 \, ds \\ &+ \sigma\int_0^t \parallel \Delta\partial_t u(s) \parallel_H^2 \, ds, \end{aligned} \tag{3.46}$$

$$I_8(t) \leqslant c_\sigma \int_0^t \parallel \Delta u(s) \parallel_H^2 \, ds + \sigma\int_0^t \parallel \Delta\partial_t u(s) \parallel_H^2 \, ds, \tag{3.47}$$

$$I_9(t) \leqslant c_\sigma \parallel \chi \parallel_{L^\infty(0,T_0;W)}^2 \int_0^t \parallel u(s) \parallel_{H^2(\Omega)}^2 \, ds + \sigma\int_0^t \parallel \Delta\partial_t u(s) \parallel_H^2 \, ds. \tag{3.48}$$

Some work has be done on the term I_{10}. First of all, Young's inequality gives

$$\begin{aligned} I_{10}(t) &\leqslant \sigma \int_0^t \| \Delta\partial_t u(s) \|_H^2 \, ds + c_\sigma \int_0^t \int_\Omega |\nabla\chi|^2 |\nabla\partial_t u||\nabla\partial_t u| \, dx \, ds \\ &=: I_{11}(t) + I_{12}(t). \end{aligned} \tag{3.49}$$

Next, we focus our attention on I_{12}: another application of Young's inequality combined with Sobolev's embeddings gives

$$\begin{aligned} I_{12}(t) &\leqslant c_\sigma \| \nabla\chi \|^2_{L^\infty(0,T_0;L^6(\Omega))} \int_0^t \| \nabla\partial_t u \|_H \| \nabla\partial_t u \|_{L^6(\Omega)} \, ds \\ &\leqslant \frac{c_\sigma \| \chi \|^2_{L^\infty(0,T_0;W)}}{2} \left(\frac{1}{\nu} \int_0^t \| \partial_t u(s) \|_V^2 \, ds \right. \\ &\quad \left. + \nu \int_0^t \| \partial_t u(s) \|^2_{H^2(\Omega)} \, ds \right). \end{aligned} \tag{3.50}$$

Now, collecting (3.45)–(3.50), choosing σ sufficiently small (depending of course on δ) and using the Poincaré inequality, one obtains

$$\begin{aligned} &\| \partial_t u(t) \|_V^2 + \delta \int_0^t \| \Delta\partial_t u(s) \|_H^2 \, ds \\ &\quad \leqslant C(\delta) \left(\| v_0 \|_V^2 + \int_0^t \| f(s) \|^2 \, ds + \left(1 + \| \chi \|^2_{L^\infty(0,T_0;W)}\right) \int_0^t \| u(s) \|^2_{H^2(\Omega)} \, ds \right. \\ &\qquad + \| \chi \|^2_{L^\infty(0,T_0;W)} \, \nu \int_0^t \| \partial_t u(s) \|^2_{H^2(\Omega)} \, ds \\ &\qquad \left. + \left(1 + \nu^{-1} \| \chi \|^2_{L^\infty(0,T_0;W)} + \| \chi \|_{L^\infty(0,T_0;W)} \right) \int_0^t \| \partial_t u(s) \|_V^2 \, ds \right). \end{aligned} \tag{3.51}$$

Now, adding to both sides the term $\delta \int_0^t \| \partial_t u(s) \|_H^2 \, ds$, exploiting well-known elliptic regularity results in order to obtain the full $L^2(0, T_0; H^2(\Omega))$-norm of u on the left side, and choosing ν sufficiently small (e.g., $\nu = \delta(2C(\delta) \| \chi \|^2_{L^\infty(0,T_0;W)})^{-1}$) we deduce (recall that $\delta \in (0, 1)$)

$$\begin{aligned} &\| \partial_t u(t) \|_V^2 + \delta \int_0^t \| \partial_t u(s) \|^2_{H^2(\Omega)} \, ds \\ &\quad \leqslant C(\delta) \left(\| v_0 \|_V^2 + \int_0^t \| f(s) \|^2 \, ds + \left(1 + \| \chi \|^2_{L^\infty(0,T_0;W)}\right) \int_0^t \| u(s) \|^2_{H^2(\Omega)} \, ds \right. \\ &\qquad \left. + \left(1 + \| \chi \|^4_{L^\infty(0,T_0;W)} + \| \chi \|_{L^\infty(0,T_0;W)} \right) \int_0^t \| \partial_t u(s) \|_V^2 \, ds \right). \end{aligned} \tag{3.52}$$

Next, we sum the inequality

$$\| u(t) \|_{H^2(\Omega)}^2 \leqslant 2 \| u_0 \|_{H^2(\Omega)}^2 + 2T_0 \int_0^t \| \partial_t u(s) \|_{H^2(\Omega)}^2 \, ds \tag{3.53}$$

to (3.52) and obtain

$$\begin{aligned} &\| \partial_t u(t) \|_V^2 + \| u(t) \|_{H^2(\Omega)}^2 + \delta \int_0^t \| \partial_t u(s) \|_{H^2(\Omega)}^2 \, ds \\ &\quad \leqslant C(\delta) \Bigg(\| v_0 \|_V^2 + \| u_0 \|_{H^2(\Omega)}^2 + \int_0^t \| f(s) \|^2 \, ds + T_0 \int_0^t \| \partial_t u(s) \|_{H^2(\Omega)}^2 \, ds \\ &\qquad + \left(1 + \| \chi \|_{L^\infty(0,T_0;W)}^2\right) \int_0^t \| u(s) \|_{H^2(\Omega)}^2 \, ds \\ &\qquad + \left(1 + \| \chi \|_{L^\infty(0,T_0;W)}^4 + \| \chi \|_{L^\infty(0,T_0;W)} \right) \int_0^t \| \partial_t u(s) \|_V^2 \, ds \Bigg). \end{aligned} \tag{3.54}$$

Thus, fixing $T_0 \leqslant \dfrac{\delta}{2C(\delta)}$, which is not restrictive since we are looking for local solutions, and using the Gronwall inequality, we conclude that

$$\| u \|_{W^{1,\infty}(0,T_0;H_0^1(\Omega)) \cap H^1(0,T_0;H^2(\Omega))} \leqslant m(R, T_0). \tag{3.55}$$

Note that here and in the sequel of this section the (possibly different) functions m depend also on δ. Moreover, a comparison argument in (2.29) gives

$$\| u \|_{H^2(0,T_0;H)} \leqslant m(R, T_0). \tag{3.56}$$

Now, thanks to (3.55) and by using standard interpolation tools (see, e.g., [11]), we can deduce an estimate for u_t in $L^{\frac{8}{3}}(0, T_0; W_0^{1,4}(\Omega))$. Thus, Hölder's inequality gives the following

$$\| u_t \|_{L^2(0,T_0;W_0^{1,4}(\Omega))}^2 \leqslant C T_0^{\frac{1}{4}} \| u_t \|_{L^{\frac{8}{3}}(0,T_0;W_0^{1,4}(\Omega))}^2 \leqslant T_0^{\frac{1}{4}} m(R, T_0). \tag{3.57}$$

Moreover, by (3.55) we get

$$\| u \|^2_{L^2(0,T_0;W_0^{1,4}(\Omega))} \leqslant T_0 m(R, T_0). \tag{3.58}$$

Thus, we obtain the following estimate for $\| u \|_{H^1(0,T_0;W_0^{1,4}(\Omega))}$:

$$\| u \|^2_{H^1(0,T_0;W_0^{1,4}(\Omega))} \leqslant \max\left\{T_0^{1/4}, T_0\right\} m_4(R, T_0). \tag{3.59}$$

Let us note that the all the constants in the estimates above, and in particular the function $m_4(R, T_0)$ in (3.59), are independent of ε. Thus, also the final time T_0 of the solution provided by Schauder's argument will not depend on ε. This will be crucial in the sequel.

Finally, we can choose T_0 in (3.59) such that

$$\max\left\{T_0^{1/4}, T_0\right\} m_4(R, T_0) \leqslant R^2, \tag{3.60}$$

to obtain

$$\| u \|^2_{H^1(0,T_0;W_0^{1,4}(\Omega))} \leqslant R^2, \tag{3.61}$$

which concludes the proof of Lemma 3.4.

3.3. Proof of Theorem 3.5

Thanks to Lemmata 3.2 and 3.4 it turns out that by choosing a proper time T_0 as in (3.60) the operator $\mathcal{S}$ is well-defined from $\mathcal{U}$ into itself. Thus, in order to prove Theorem 3.5, i.e., to apply the Schauder fixed-point Theorem, we need to verify (3.14) and (3.15). Regarding (3.15), we observe that, given $\delta, R > 0$ and choosing T_0 as in (3.60), we derive from (2.5) and (3.7) that, $\forall \bar{u} \in \mathcal{U}$, the corresponding $u = \mathcal{S}(\bar{u})$ satisfies

$$\| u \|_{H^1(0,T_0;H^2(\Omega))\cap W^{1,\infty}(0,T_0;H_0^1(\Omega))} \leqslant C, \tag{3.62}$$

for a constant C not depending on $\bar{u}$. Hence, by a comparison of the terms in (2.29) it is not difficult to infer

$$\| \partial_{tt} u \|_{L^2(0,T_0;H)} \leqslant C. \tag{3.63}$$

Thus, the latter two estimates combined with (3.57) and Sobolev's embeddings Theorems guarantee that $\mathcal{S}$ is a compact operator, i.e., (3.15) holds. Indeed, for $\tau > 0$ we have $H^2(0,T_0;H)\cap H^1(0,T_0;H^2(\Omega)) \subset\subset H^1(0,T_0;H^{2-\tau}(\Omega)) \subset H^1(0,T_0;W^{1,4}(\Omega))$, the latter inclusion actually holding for sufficiently small τ. Finally, we aim to show that $\mathcal{S}$ is continuous with respect to the natural strong topology induced on $\mathcal{U}$ by $H^1(0,T_0;W_0^{1,4}(\Omega))$, i.e., property (3.14). Thus, given a sequence

$$\bar{u}_n \to \bar{u} \quad \text{strongly in } \mathcal{U}, \tag{3.64}$$

we aim to study the behavior of $\mathcal{S}(\bar{u}_n)$ as $n \nearrow +\infty$ and, in particular, to prove that

$$\mathcal{S}(\bar{u}_n) \to \mathcal{S}(\bar{u}) \quad \text{strongly in } \mathcal{U}. \tag{3.65}$$

As a first step, let us consider the sequence of solutions to Problem 3.1 obtained once $\bar{u}_n$ substitutes $\bar{u}$, i.e., $\mathcal{S}_1(\bar{u}_n) = \chi_n$. Recalling (3.7), there holds the following bound (with the constant C independent of n)

$$\| \chi_n \|_{H^1(0,T_0;V)\cap L^\infty(0,T_0;W)} \leqslant C, \tag{3.66}$$

which allows us to extract a subsequence (not relabeled) of n such that

$$\chi_n \to \chi \quad \text{weakly star in } H^1(0,T_0;V) \cap L^\infty(0,T_0;W), \tag{3.67}$$

for some suitable function χ. Moreover, using again the compactness result in [12, Corollary 4], we have the following strong convergence:

$$\chi_n \to \chi \quad \text{strongly in } C^0([0,T_0];V). \tag{3.68}$$

Concerning now the right-hand side of Eq. (3.17), we have

$$-|\nabla\bar{u}_n|^2 \to -|\nabla\bar{u}|^2 \quad \text{strongly in } H^1(0,T_0;H). \tag{3.69}$$

Then, reproducing the same argument exploited in the proof of Lemma 3.2, it is not difficult to show that χ is the solution to Problem 3.1 corresponding to the limit datum $\bar{u}$. Moreover, the uniqueness property of Lemma 3.2 gives that the whole sequence χ_n converges to χ; thus, eventually we have that $\chi = \mathcal{S}_1(\bar{u})$. As a second step, let us set $\chi_n = \mathcal{S}_1(\bar{u}_n)$ in (2.29) and consider the corresponding solutions u_n. By performing the same estimates as in the proof of Lemma 3.4, we have that (3.55)–(3.56) hold independently of n. This leads to the extraction of a subsequence of n such that

$$u_n \to u \quad \text{weakly star in } H^1(0,T_0;H^2(\Omega)) \cap W^{1,\infty}(0,T_0;H_0^1(\Omega))$$

and

$$\partial_{tt}u_n \to \partial_{tt}u \quad \text{weakly in } L^2(0,T_0;H), \tag{3.70}$$

for a suitable limit function u. The latter weak convergences and the same argument used before to prove compactness of $\mathcal{S}$ give then

$$u_n \to u \quad \text{strongly in } H^1(0,T_0;W_0^{1,4}(\Omega)). \tag{3.71}$$

Now, combining (3.68) with (3.70) and recalling the definition of T_δ, it is not difficult to see that we can pass to limit in (2.29) (written for u_n and χ_n) as $n \nearrow +\infty$. Indeed, at the limit, we get that the equation is solved by u and $\chi = \mathcal{S}_1(\bar{u})$. Thus, the uniqueness part of Lemma 3.4 guarantees that (3.70)–(3.71) hold for the whole sequence u_n, so that $u = \mathcal{S}_2(\chi)$. Finally, (3.68) and (3.71) give (3.65) or, equivalently (3.14), which concludes the proof.

3.4. Conclusion of the proof of Theorem 2.5

As a final step, we have to pass to the limit as $\varepsilon \searrow 0$. Thus, it is convenient to come back to the notations χ^ε, etc., in the sequel. Then, since all the preceding estimates hold independently of ε, we have limit functions χ, η, u such that, at least for a subsequence (not relabeled)

$$\chi^\varepsilon \to \chi \quad \text{weakly star in } H^1(0,T_0;V) \cap L^\infty(0,T_0;W), \tag{3.72}$$

$$\beta^\varepsilon(\chi^\varepsilon) \to \eta \quad \text{weakly star in } L^\infty(0,T_0;H), \tag{3.73}$$

$$u^\varepsilon \to u \quad \text{weakly star in } H^1(0,T_0;H^2(\Omega)) \cap W^{1,\infty}(0,T_0;H_0^1(\Omega)), \tag{3.74}$$

$$\partial_{tt}u_\varepsilon \to \partial_{tt}u \quad \text{weakly in } L^2(0,T_0;H). \tag{3.75}$$

Now, while the passage to the limit as $\varepsilon \searrow 0$ in the equation for u (2.29) can be performed exactly as in the preceding step, we have to take some more care in dealing with Eq. (2.26). Indeed, convergences (3.72),(3.74)–(3.75) combined with [12, Corollary 4] give

$$\chi^\varepsilon \to \chi \quad \text{strongly in } C^0([0,T_0];V), \tag{3.76}$$

$$u^\varepsilon \to u \quad \text{strongly in } H^1(0,T_0;W_0^{1,4}(\Omega)) \tag{3.77}$$

and this, together with (3.73), entails the immediate identification $\eta \in \beta(\chi)$. Moreover, arguing as for (3.36), we can prove that

$$\zeta^\varepsilon \to \zeta \quad \text{weakly star in } L^\infty(0,T_0;H), \quad \text{where } \zeta^\varepsilon := \partial_t\chi^\varepsilon + \xi^\varepsilon, \tag{3.78}$$

for some limit function ζ. This allows us to pass to the limit in the equation. As before, in order to interpret ζ in terms of α, we exploit a semicontinuity-comparison tool. Namely, we aim to prove that

$$\limsup_{\varepsilon \searrow 0} \int_0^{T_0} \big(\zeta^\varepsilon, \partial_t \chi^\varepsilon\big)(t)\, dt \leqslant \int_0^{T_0} \big(\zeta, \partial_t \chi\big)(t)\, dt. \tag{3.79}$$

Then, we exploit the argument given in (3.37)–(3.39), but no longer treat β as in (3.39). Indeed, integrating by parts in time, it is now enough to show

$$\liminf_{\varepsilon \searrow 0} \int_\Omega \phi^\varepsilon(\chi^\varepsilon(T_0)) \geqslant \int_\Omega \phi(\chi(T_0)). \tag{3.80}$$

This easily follows since the functional induced by ϕ_ε on $L^2(\Omega)$ (cf. (2.10)) converges in the sense of Mosco [1, Proposition 3.56, p. 354] to the functional

$$\Phi(v) := \begin{cases} \int_\Omega \phi(v) & \text{if } \phi(v) \in L^1(\Omega), \\ +\infty & \text{otherwise} \end{cases} \tag{3.81}$$

and we have

$$\chi^\varepsilon(T_0) \to \chi(T_0) \quad \text{strongly in } L^2(\Omega). \tag{3.82}$$

4. Proof of Theorem 2.2

In this section we aim to prove that, at least up to a small final time $\hat{t}$ to be specified, the quadruple (χ, ξ, η, u) solving (2.26)–(2.30) actually solves (2.16)–(2.20). In particular, to show property (2.15), which actually says that the truncation has no effect, it is sufficient to find a time $\hat{t}$ such that

$$\| \chi - 1 \|_{L^\infty(\Omega\times(0,\hat{t}))} \leqslant 1 - \delta. \tag{4.1}$$

Now, from (2.22) and (3.7), we infer that there exists $m_5(R, T_0)$ such that

$$\| \chi(t) - 1 \|_W \leqslant m_5(R, T_0) \text{ and}$$

$$\| \chi(t) - 1 \|_V \leqslant t^{1/2} \| \partial_t \chi \|_{L^2(0,t;V)} \leqslant m_5(R, T_0) t^{1/2}, \quad \forall t \in [0, T_0], \tag{4.2}$$

where we remark once more that the above function m_5 is effectively computable in functions of the data. Next, following the notations of Lions and Magenes

[10, Theorem 9.6, p. 43], we consider the interpolation space

$$H^{\frac{5}{3}}(\Omega) = \left[H^2(\Omega), H^1(\Omega)\right]_{\frac{1}{3}}, \tag{4.3}$$

which is continuously embedded into $L^\infty(\Omega)$ in our three dimensional setting (see, e.g., [10, Theorem 9.8, p. 45]). Thus, the interpolation inequality, the previous immersion, and (4.2) entail

$$\begin{aligned} \| \chi(t) - 1 \|_{L^\infty(\Omega)} &\leqslant C \| \chi(t) - 1 \|^{2/3}_{H^2(\Omega)} \| \chi(t) - 1 \|^{1/3}_V \\ &\leqslant Cm_5(R, T_0) t^{1/6}, \quad \forall t \in [0, T_0]. \end{aligned} \tag{4.4}$$

Thus, fixing $\hat{t} \in [0, T_0]$ such that

$$\hat{t} \leqslant \left(\frac{1-\delta}{Cm_5(R, T_0)}\right)^6, \tag{4.5}$$

where C is the embedding constant in (4.4), it is straightforward to check that (2.15) holds, which concludes the proof of Theorem 2.2.

5. Proof of Theorem 2.4

In this section we outline the proof of Theorem 2.4. To this aim, we consider the couple of solutions to (2.11)–(2.20) introduced in the statement and set $(\chi, \xi, \eta, u) := (\chi_1 - \chi_2, \xi_1 - \xi_2, \eta_1 - \eta_2, u_1 - u_2)$. Then, we take the difference between (2.19) written for (u_1, χ_1) and for (u_2, χ_2), test it by $\partial_t u$ and integrate in time up to $t \leqslant \hat{t}$, where $\hat{t}$ is the reference time introduced in the statement. Recalling (2.15), it is not difficult to infer

$$\begin{aligned} &\frac{1}{2} \| \partial_t u(t) \|^2_H + \delta \int_0^t \| \nabla \partial_t u(s) \|^2_H \, ds \\ &\quad \leqslant \int_0^t \int_\Omega |\chi_1(s)| \cdot |\nabla u(s)| \cdot |\nabla \partial_t u(s)| \, dx \, ds + \int_0^t \int_\Omega |\chi(s)| \big(|\nabla u_2(s)| \\ &\qquad + |\nabla \partial_t u_2(s)|\big) |\nabla \partial_t u(s)| \, dx \, ds =: I_{13}(t) + I_{14}(t). \end{aligned} \tag{5.1}$$

Since $|\chi_1(x,t)| < 1$ for almost any $(x, t) \in \Omega \times (0, \hat{t})$, and thanks to the third of (2.14), the integrals $I_{13}(t), I_{14}(t)$ can be estimated in this way:

$$I_{13}(t) \leqslant c_\delta \int_0^t \| \nabla u(s) \|^2_H \, ds + \frac{\delta}{4} \int_0^t \| \nabla \partial_t u(s) \|^2_H \, ds, \tag{5.2}$$

$$I_{14}(t) \leqslant \frac{\delta}{4}\int_0^t \| \nabla\partial_t u(s) \|_H^2 \, ds + c_\delta \int_0^t \| \big(\nabla u_2(s) + \nabla\partial_t u_2(s)\big)(s) \|_V^2 \times \| \chi(s) \|_V^2 \, ds. \tag{5.3}$$

Adding now

$$\frac{\delta}{4\hat{t}} \| \nabla u(t) \|_H^2 \leqslant \frac{\delta}{4}\int_0^t \| \nabla\partial_t u(s) \|_H^2 \, ds, \quad t \in (0, \hat{t}),$$

to the resulting inequality, where $\hat{t}$ is the life time of our problem, we obtain

$$\begin{aligned} &\| \partial_t u(t) \|_H^2 + \| \nabla u(t) \|_H^2 + \int_0^t \| \nabla\partial_t u(s) \|_H^2 \, ds \\ &\leqslant C(\delta, \hat{t})\left(\int_0^t \| \nabla u(s) \|_H^2 \, ds + \int_0^t \| \big(\nabla u_2(s) + \nabla\partial_t u_2(s)\big)(s) \|_V^2 \right. \\ &\quad \left. \times \| \chi(s) \|_V^2 \, ds\right). \end{aligned} \tag{5.4}$$

Next, we write (2.16) firstly for $(\chi_1, \xi_1, \eta_1, u_1)$, then for $(\chi_2, \xi_2, \eta_2, u_2)$, take the difference, test by $\partial_t\chi$, and integrate over $(0, t)$ with $t \leqslant \hat{t}$. We get

$$\begin{aligned} &\int_0^t \| \partial_t\chi(s) \|_H^2 \, ds + \tfrac{1}{2} \| \nabla\chi(t) \|_H^2 \\ &\leqslant -\int_0^t (\beta(\chi_1(s)) - \beta(\chi_2(s)), \partial_t\chi(s))\, ds \\ &\quad -\frac{1}{2}\int_0^t \left(|\nabla u_1(s)|^2 - |\nabla u_2(s)|^2, \partial_t\chi(s)\right) ds =: I_{15}(t) - I_{16}(t). \end{aligned} \tag{5.5}$$

Our next aim is clearly to provide a bound for I_{15} and I_{16}. As for I_{15}, being β Lipschitz, we have

$$I_{15}(t) \leqslant \frac{1}{2}\int_0^t \| \partial_t\chi(s) \|_H^2 \, ds + c\int_0^t \| \chi(s) \|_V^2 \, ds. \tag{5.6}$$

Concerning I_{16}, an integration by parts with respect to time gives

$$\begin{aligned} I_{16}(t) &= \frac{1}{2}\left(|\nabla u_1(t)|^2 - |\nabla u_2(t)|^2, \chi(t)\right) \\ &\quad - \int_0^t \left(\big(\nabla u(s), \chi(s)\nabla\partial_t u_1(s)\big) + \big(\nabla\partial_t u(s), \chi(s)\nabla u_2(s)\big)\right) ds. \end{aligned} \tag{5.7}$$

Thus, recalling (2.14), it is not difficult to infer

$$\begin{aligned} I_{16}(t) \leqslant\ & c_{\varepsilon_1} \| \nabla u(t) \|_H^2 + \varepsilon_1 \| \chi(t) \|_V^2 \\ & + C \left(\int_0^t \| \nabla u(s) \|_H^2 \, ds + \int_0^t \| \partial_t u_1(s) \|_{H^2(\Omega)}^2 \| \chi(s) \|_V^2 \, ds \right) \\ & + \varepsilon_2 \int_0^t \| \nabla \partial_t u(s) \|_H^2 + c_{\varepsilon_2} \int_0^t \| u_2(s) \|_{H^2(\Omega)}^2 \| \chi(s) \|_V^2 \, ds, \end{aligned} \tag{5.8}$$

where the constant c_{ε_1} depends also on the norms

$$\| u_1 \|_{L^\infty(0,\hat{t},H^2(\Omega))}, \quad \| u_2 \|_{L^\infty(0,\hat{t},H^2(\Omega))},$$

which are bounded quantities owing to third of (2.14). Now, let us add to the resulting (5.5) the inequality

$$\frac{1}{4\hat{t}} \| \chi(t) \|_H^2 \leqslant \frac{1}{4} \int_0^t \| \partial_t \chi(s) \|_H^2 \, ds, \quad \text{holding } \forall t \in [0, \hat{t}].$$

Then, choosing $\varepsilon_1 < \min \left\{ \frac{1}{2}, \frac{1}{4\hat{t}} \right\}$ and combining (5.6)–(5.8), inequality (5.5) becomes

$$\begin{aligned} & \int_0^t \| \partial_t \chi(s) \|_H^2 \, ds + \| \chi(t) \|_V^2 \\ & \leqslant C(\hat{t}) \Bigg(\| \nabla u(t) \|_H^2 + \varepsilon_2 \int_0^t \| \nabla \partial_t u(s) \|_H^2 \, ds + \int_0^t \| \nabla u(s) \|_H^2 \, ds \\ & \quad + \int_0^t \left(1 + \| \partial_t u_1(s) \|_{H^2(\Omega)}^2 + c_{\varepsilon_2} \| u_2(s) \|_{H^2(\Omega)}^2 \right) \| \chi(s) \|_V^2 \, ds \Bigg). \end{aligned} \tag{5.9}$$

Finally, adding (5.9)–(5.4) multiplied by a proper scaling constant $M > C(\hat{t})$, choosing ε_2 small enough (i.e., $\varepsilon_2 < M/C(\hat{t})$), we get

$$\begin{aligned} & \| \partial_t u(t) \|_H^2 + \| \nabla u(t) \|_H^2 + \int_0^t \| \nabla \partial_t u(s) \|_H^2 \, ds + \int_0^t \| \partial_t \chi(s) \|_H^2 \, ds + \| \chi(t) \|_V^2 \\ & \leqslant C(\delta, \hat{t}, M, \varepsilon_2) \Bigg(\int_0^t \| \nabla u(s) \|_H^2 \, ds + \int_0^t \Big(1 + \| u_2(s) \|_{H^2(\Omega)}^2 \\ & \quad + \| \partial_t u_2(s) \|_{H^2(\Omega)}^2 + \| \partial_t u_1(s) \|_{H^2(\Omega)}^2 \Big) \| \chi(s) \|_V^2 \, ds \Bigg), \end{aligned} \tag{5.10}$$

where $\| u_2 \|^2_{H^2(\Omega)}$ and $\| \partial_t u_i \|^2_{H^2(\Omega)}$, $i = 1, 2$, belong to $L^1(0, T_0)$. Thus, we can apply Gronwall's Lemma to (5.10) to conclude the proof.

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